Study on Delaunay tessellations of 1-irregular cuboids for 3D mixed element meshes

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Summary. Mixed elements meshes based on the modified octree approach contain several co-spherical point configurations. While generating Delaunay tessellations to be used together with the finite volume method, it is not necessary to partition them into tetrahedra; co-spherical elements can be used as final elements. This paper presents a study of all co-spherical elements that appear while tessellating a 1-irregular cuboid (cuboid with at most one Steiner point on its edges) with different aspect ratio. Steiner points can be located at any position between the edge endpoints. When Steiner points are located at edge midpoints, 24 co-spherical elements appear while tessellating 1-irregular cubes. By inserting internal faces and edges to these new elements, this number is reduced to 13. When 1-irregular cuboids with aspect ratio equal to $\sqrt{2}$ are tessellated, 10 co-spherical elements are required. If 1-irregular cuboids have aspect ratio between 1 and $\sqrt{2}$, all the tessellations are adequate for the finite volume method. When Steiner points are located at any position, the study was done for a specific Steiner point distribution on a cube. 38 co-spherical elements were required to tessellate all the generated 1-irregular cubes. Statistics about the impact of each new element in the tessellations of 1-irregular cuboids are also included. This study was done by developing an algorithm that construct Delaunay tessellations by starting from a Delaunay tetrahedral mesh built by Qhull.

1 Introduction

Scientific and engineering problems are usually modeled by a set of partial differential equations and the solution to these partial differential equations is calculated through the use of numerical methods. In order to get good results, the object being modeled (domain) must be discretized in a proper way respecting the requirements imposed by the used numerical method. The discretization (mesh) is usually composed of simple cells (basic elements) that must represent the domain in the best possible way. In particular, we are interested in meshes for the finite volume method which are formed by polygons (in a 2D domain) or polyhedra (in a 3D domain), that satisfy the Delaunay

condition: the circumcircle in 2D, or circumsphere in 3D, of each element does not contain any other mesh point in its interior. The Delaunay condition is required because we use its dual structure, the Voronoi diagram, to model the control volumes in order to compute an approximated solutions. The basic elements used so far are triangles and quadrilaterals in 2D, and tetrahedra, cuboids, prisms and pyramids in 3D. Meshes composed of different elements types are called mixed element meshes. [1]
Mixed element meshes are built on 2D or 3D domains described by sets of points, polygons or polyhedra depending on the application. We have developed a mixed element mesh generator [2] based on an extension of octrees [3, 4]. Our approach starts enclosing the domain in the smallest bounding box (cuboid). Second, this cuboid is continuously refined, at any edge position, by using the geometry information of the domain. That is why this refinement is called intersection based approach. Once this step finishes, an initial non-conforming mesh composed of tetrahedra, pyramids, prisms, and cuboids is generated that fits the domain geometry. Third, these elements are further refined by bisection, as far as possible, until the density requirements are fulfilled. Fourth, the mesh is done 1-irregular by allowing only one Steiner point on each edge. The current solution is based on patterns but only the most frequently used patterns are available. Then, if a pattern is not available or the element can not be properly tessellated for the finite volume method, new Steiner points are inserted until all 1-irregular elements can be properly tessellated. The current set of seven final elements is shown in Figure 1.

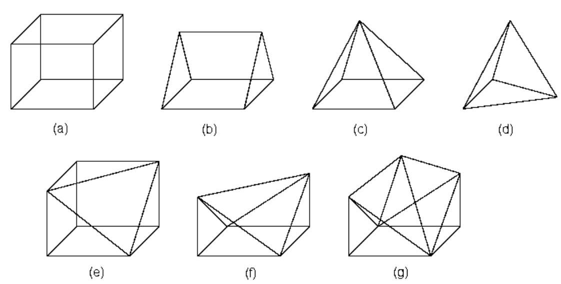

Fig. 1. The seven final elements of the Ω Mesh Generator: (a) Cuboid, (b) Triangular Prism, (c) Quadrilateral Pyramid, (d) Tetrahedron, (e) Tetrahedron Complement, (f) Deformed Prism, and (g) Deformed Tetrahedron Complement.

The advantage of using a mixed mesh in comparison with a tetrahedral mesh is that the use of different element types reduce the amount of edges, faces and elements in the final mesh. For example, we do not need to divide a cuboid into tetrahedra. On the other hand, a disadvantage is that the equations must be discretized using different elements.
Octree based approaches naturally produces co-spherical point sets. A mixed

mesh satisfying the Delaunay condition can include all produced co-spherical elements as shown in Figure 2. The final elements in this example are five pyramids and four tetrahedra.

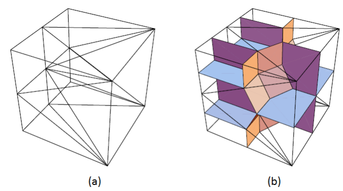

Fig. 2. (a) Mixed mesh of a 1-irregular cuboid that satisfies the Delaunay condition, (b) the same mixed mesh and its associated Voronoi diagram.

The goal of this paper is to study the co-spherical elements that can appear while tessellating 1-irregular cuboids generated by using a bisection and intersection based approach and to analyze how useful would be to include the new elements in the final element set. In particular, this paper gives the number and shape of new co-spherical elements needed to tessellate (a) all 1-irregular cuboids generated by the bisection approach and (b) some particular 1-irregular cuboids generated by the intersection refinement approach. In addition, statistics associated with particular tessellations are presented such as the frequency each co-spherical element is used and the number of tessellations that can be used with the finite volume method without adding extra vertices. The analysis of the 1-irregular cuboid tessellations was done under different criteria that affect the amount of generated co-spherical elements.
We have focused this work on the analysis of the tessellations of 1-irregular cuboids because this element is the one that more frequently appears when meshes are generated by a modified octree approach. A theoretical study on the number of different 1-irregular cuboid configurations that can appear either by using a bisection or an intersection based approach was published in [5]. We use the results of that work as starting point for this study.
This paper is organized as follows: Section 2 describes the bisection and intersection refinement approaches. Section 3 presents briefly the developed algorithm to compute Delaunay tessellations. Section 4 and Section 5 give the results obtained by applying the algorithm to 1-irregular cuboids generated by a bisection and an intersection based approach, respectively. Section 6 includes our conclusions.

2 Basic concepts

This section describes some ideas in order to understand how 1-irregular cuboids are generated.

Bisection based approach 1-irregular configurations

In this approach, the Steiner points inserted at the refinement phase are always located at the edge midpoints. Cuboids can be refined into two, four or eight smaller cuboids as shown in Figure 3.

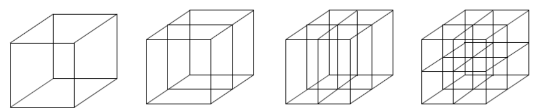

Fig. 3. Cuboid and its splits into two, four and eight cuboids using a bisection based approach.

This refinement produces neighboring cuboids with Steiner points located at the edge midpoints. Those 1-irregular cuboids are larger than the already refined neighbor cuboid as shown in Figure 4.

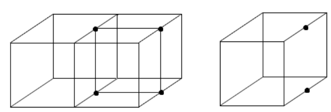

Fig. 4. The bisection-refined cuboid at the left produces an 1-irregular element like the cuboid at the right

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Intersection based approach 1-irregular configurations

While using an intersection based approach, the Steiner points are not necessarily located at the edge midpoints. In general, there are no constrains on the location of the Steiner points, except by the fact that parallel edges must be divided in the same relative position to ensure the generation of cuboids and not any other polyhedron. Figure 5 shows an example of this approach.

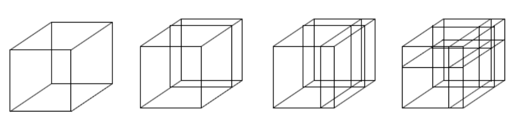


Fig. 5. Cuboid and its splits into two, four and eight cuboids using an intersection based approach.

This refinement produces neighboring cuboids with Steiner points located at any edge position. Those 1-irregular cuboids are larger than the already refined neighbor cuboid as shown in Figure 4.

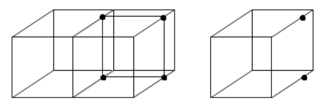


Fig. 6. The intersection-refined cuboid to the left produces an 1-irregular element like the cuboid to the right.

3 Algorithm

In order to count the number of new co-spherical elements than can appear and to recognize their shape, we have developed an algorithm that executes the followings steps:

1. Build the point configuration of a 1-irregular cuboid by specifying the coordinates of the cuboid vertices and its Steiner points.
2. Build a Delaunay tetrahedral mesh for this point configuration by using QHull[1].
3. Join tetrahedra to form the largest possible co-spherical elements.
4. Identify each final co-spherical polyhedron.

Qhull divides co-spherical point configurations into a set of tetrahedra by adding an artificial point that is not part of the input. Then, we use this fact to recognize the faces that form a co-spherical polyhedron and later to recognize which element is.

4 Results: Bisection based approach

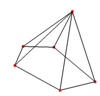
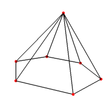
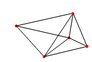
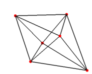
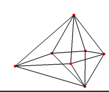
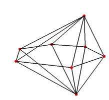
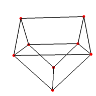
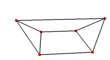
This section describes the results obtained by applying the previous algorithm to the 4096 (2^{12}) 1-irregular configurations that can be generated using a bisection based approach. First, the new co-spherical elements are shown. Then, their impact in all the tessellations is analyzed and finally, the tessellations that can be used with the finite volume method are characterized.

[1]http://www.qhull.org

4.1 New co-spherical elements

We have identified 17 new co-spherical polyhedra in the tessellations of 1-irregular cubes in addition to the seven original elements shown in Figure 1. A description of each one can be found in Table 1. A distinction is made between rectangular and quadrilateral faces except for the quadrilateral pyramid. Because of this, the triangular prism and the generic element #1 are considered different co-spherical elements, and the same happens between the deformed prism and the generic element # 3. This could be changed in a future study.

Table 1. Description of the new co-spherical elements that appear while tessellating 1-irregular cubes

Element	**Vertices**	**Edges**	**Faces**	**Example**
Pentagonal Pyramid	6	10	6	
Hexagonal Pyramid	7	12	7	
Triangular Bipyramid	5	9	6	
Quadrilateral Bipyramid	6	12	8	
Pentagonal Bipyramid	7	15	10	
Hexagonal Bipyramid	8	18	12	
Triangular Biprism	8	14	8	
Generic #1	6	9	5	

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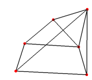
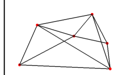
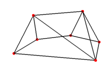
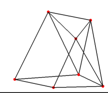
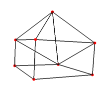
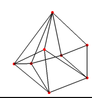
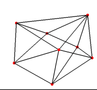
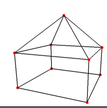
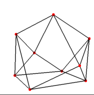
Element	**Vertices**	**Edges**	**Faces**	**Example**
Generic #2	6	10	6	
Generic #3	6	11	7	
Generic #4	7	12	7	
Generic #5	7	13	8	
Generic #6	8	15	9	
Generic #7	8	16	10	
Generic #8	8	17	11	
Generic #9	9	16	9	
Generic #10	9	18	11	

4.2 Element analysis

Since there are 17 new co-spherical elements, the natural question is if this number can be reduced without adding diagonals in the cuboid rectangular faces. In fact, our mixed element mesh generator requires to tessellate 1-irregular cuboids without adding diagonals on its rectangular faces when it uses a pattern-wise approach. In the following, we analyze the number of co-spherical elements under three different criteria:

- **Finding the optimal tessellation**: An optimal tessellation contains the lowest amount of final elements. This is reached by maximizing the number of elements with different shape. The number of co-spherical elements that can be used is 24.

- **Minimizing the number of different co-spherical elements by adding only internal faces**: Under this criterion we want to reduce the number of different final elements by adding only internal faces. Examining the set of new elements in Table 1, we see that the bipyramids and the biprisms are naturally divisible into two elements, and so are the generic #5 (separable into a prism and a quadrilateral pyramid), generic #8 (separable into a prism and two quadrilateral pyramids) and generic #9 (separable into a cuboid and quadrilateral pyramid), among others. An example of this type of separation is shown in Figure 7. The total number of co-spherical elements needed to tessellate the 4096 configurations is now 16.

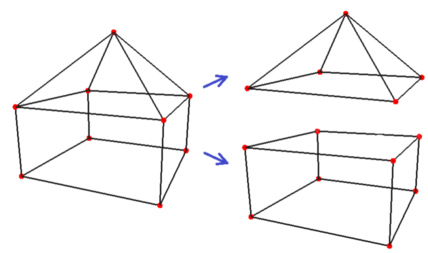


Fig. 7. Generic #9 element and its separation into two different elements.

- **Minimizing the number of different co-spherical elements by adding internal edges and faces**: This extends the second criterion by adding the condition that it is possible to add extra edges only if they are inside the new elements. The reason for only allowing internal edges is that adding external edges could change the partition of one of the rectangular faces of the original cuboid. Under this criterion, the elements that are separable are generic #3 (one inner edge produces two tetrahedra and one quadrilateral pyramid), generic #6 (one inner edge produces two tetrahedra and a tetrahedron complement) and generic #7 (two inner edges produce two tetrahedra, a quadrilateral pyramid and a deformed prism). An example of this type of separation is shown in Figure 8. The total number of co-spherical elements needed to tessellate the 4096 configurations is reduced to 13.

4.3 Evaluating the impact of each new element

In order to decide how important is to include a new element in the final element set, in this section we study how many times each co-spherical element appears in the tessellation of a 1-irregular cuboid. For this study, we have run our program for 1-irregular cuboids with three different aspect ratio: 1, 4, and $\sqrt{2}$.

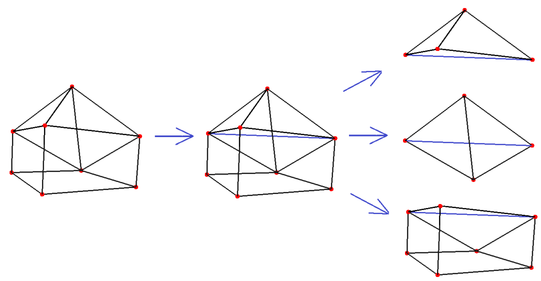


Fig. 8. Generic #6 element and its separation into three different elements by adding an extra inner edge shown in blue.

- **Test case A**. Aspect ratio equal to 1 ($a = b = c$): The 1-irregular cube appears naturally on the standard octree and this method is used by most mesh generators based on octrees.
- **Test case B**. Aspect ratio equal to 4 ($4a = 2b = c$): This represents a typical cuboid to model thin zones.
- **Test case C**. Aspect ratio equal to $\sqrt{2}$ ($a\sqrt{2} = b = c$): It was shown in [6] that some 1-irregular cuboid within these proportions can be tessellated without problems for the finite volume method.

Running the test case A

Table 2 shows the frequency in which appear each one of the 24 co-spherical elements in the tessellations of 1-irregular cubes.

Table 2. Frequency of the co-spherical elements on 1-irregular cube tessellations

Element	Freq.	Element	Freq.
Cuboid	195	Hexagonal Bipyramid	36
Tetrahedron	18,450	Triangular Biprism	6
Quadrilateral Pyramid	11,718	Generic #1	12
Triangular Prism	3,720	Generic #2	96
Tetrahedron Comp.	992	Generic #3	48
Def. Prism	396	Generic #4	48
Def. Tetrahedron Comp.	144	Generic #5	120
Pentagonal Pyramid	384	Generic #6	24
Hexagonal Pyramid	56	Generic #7	48
Triangular Bipyramid	240	Generic #8	48
Quadrilateral Bipyramid	272	Generic #9	6

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Element	Freq.	Element	Freq.
Pentagonal Bipyramid	192	Generic #10	8
		Total	**37,259**

From Table 2, we observe that the most used elements correspond to tetrahedra and quadrilateral pyramids (∼49.5% and ∼31.5% of the total elements, respectively). Moreover, the set of seven initial co-spherical elements represents ∼95.6% of the total. If the number of co-spherical elements is reduced to 16 by adding internal faces, the element frequencies are distributed as shown in Table 3. It can be observed that the most used elements are again tetrahedra and quadrilateral pyramids (∼49.5% and ∼32.7% of elements, respectively). The set of seven initial co-spherical elements represents ∼96.9% of the total.

Table 3. Frequency of the co-spherical elements adding only internal faces on 1-irregular cube tessellations

Element	Freq.	Element	Freq.
Cuboid	201	Hexagonal Pyramid	128
Tetrahedron	18,930	Generic #1	12
Quadrilateral Pyramid	12,484	Generic #2	96
Triangular Prism	3,900	Generic #3	48
Tetrahedron Comp.	992	Generic #4	48
Def. Prism	396	Generic #6	24
Def. Tetrahedron Comp.	144	Generic #7	48
Pentagonal Pyramid	768	Generic #10	8
		Total	**38,227**

Finally, when the number of different final co-spherical elements is reduced to 13, by adding internal edges and faces, the element frequencies are shown in Table 4. The most used elements correspond to tetrahedra and quadrilateral pyramids (∼49.8% and ∼32.7% of the total number of elements, respectively). The set of initial seven co-spherical elements represents ∼97.2% of the total.

Table 4. Frequency of co-spherical elements adding internal faces and edges on 1-irregular cube tessellations

Element	Freq.	Element	Freq.
Cuboid	201	Pentagonal Pyramid	768
Tetrahedron	19,170	Hexagonal Pyramid	128
Quadrilateral Pyramid	12,580	Generic #1	12
Triangular Prism	3,900	Generic #2	96
Tetrahedron Comp.	1,016	Generic #4	48

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Element	Freq.	Element	Freq.
Def. Prism	444	Generic #10	8
Def. Tetrahedron Comp.	144		
		Total	**38,515**

Running the test Case B

When the aspect ratio of the cuboid is changed to 4, only 6 different co-spherical elements appear and their frequencies are shown in Table 5.

Table 5. Frequency of the co-spherical elements in the tessellations of 1-irregular cuboids with aspect ratio 4

Element	Freq.	Element	Freq.
Cuboid	103	Triangular Prism	3,120
Tetrahedron	29,118	Tetrahedron Comp.	536
Quadrilateral Pyramid	12,620	Def. Prism	84
		Total	**45,581**

From Table 5, we observe that tetrahedra and quadrilateral pyramids are the most used elements, comprising more than 90% of the total of the elements (∼63.9% of tetrahedra and ∼27.7% quadrilateral pyramids). Note that these elements can not be divided into simpler ones without adding diagonals on its quadrilateral faces.

Running the test Case C

When the aspect ratio is equal to $\sqrt{2}$, only 10 different final co-spherical elements appear whose frequencies are distributed as follows:

Table 6. Frequency of the final elements on optimal tessellations on 1-irregular cuboids with aspect ratio $\sqrt{2}$

Element	Freq.	Element	Freq.
Cuboid	199	Def. Prism	284
Tetrahedron	25,252	Triangular Bipyramid	128
Quadrilateral Pyramid	12,300	Quadrilateral Bipyramid	52
Triangular Prism	3,780	Generic #2	16
Tetrahedron Comp.	1,008	Generic #5	16
		Total	**43,035**

The most used elements correspond to tetrahedra and quadrilateral pyramids (∼58.7% and ∼28.6% of the total number of elements, respectively). Moreover, the set of initial seven co-spherical elements represents a ∼99.5% of the total.

In this test case, the number of co-spherical elements can be reduced from 10 to 7 by adding internal faces. The frequencies of these seven elements are distributed as shown in Table 7.

Table 7. Frequency of the 7 co-spherical elements while tessellating 1-irregular cuboids with aspect ratio $\sqrt{2}$

Element	Freq.	Element	Freq.
Cuboid	199	Tetrahedron Comp.	1,008
Tetrahedron	25,508	Def. Prism	284
Quadrilateral Pyramid	12,420	Generic #2	16
Triangular Prism	3,796		
		Total	**43,231**

Again, the most used elements are tetrahedra and quadrilateral pyramids ($\sim$59.0% and $\sim$28.7% of elements, respectively). There are only 6 of the seven initial co-spherical elements, representing a $\sim$99.96% of the total. Notice that this set of 7 elements is not separable by adding internal edges or faces.

4.4 Tessellations and the finite volume method

We have also examined whether the generated tessellations meet the requirements for their use in the context of the finite volume method. The requirement is that the circumcenter of each final element is contained within the initial 1-irregular cuboid. This requirement is strong but it allows our mesh generator to find a proper tessellation of each 1-irregular cuboid locally.
The evaluation of each tessellation is performed on the same test cases discussed in Section 4.2. The results are shown in Table 8. We observe that the circumcenters of all elements are inside the initial cuboid for all the configurations in the test cases A and C. This means that all 1-irregular configurations could be properly tessellated if the aspect ratio of the elements is less or equal to $\sqrt{2}$. If 1-irregular cuboids has an aspect ratio equal to 4, only 132 1-irregular cuboids fit the circumcenter requirement.

Table 8. Number of configurations that fit the circumcenter requirement

	Number of proper configurations
Test Case A (Aspect ratio equal to 1)	4096
Test Case B (Aspect ratio equal to 4)	132
Test Case C (Aspect ratio equal to $\sqrt{2}$)	4096

5 Results: Intersection based approach

The number of 1-irregular configurations that can appear while refining cuboids by an intersection based approach is $187^3 + 1$ [5]. We consider that two 1-irregular configurations are different if the relative position of Steiner points located on parallel cuboid edges is not the same. In this section we describe the results obtained by applying the algorithm to all 1-irregular configurations of a cube that can be generated by inserting Steiner points only on the positions defined by multiples of 1/16 of the edge length. The impact of each co-spherical element was only obtained for the 1-irregular cube.

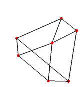
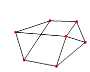
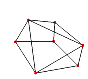
5.1 New co-spherical elements

Since the possible positions of Steiner points on a particular edge are infinite, we only use a set of predetermined Steiner point positions for each set of cuboid parallel edges defined as follows:

- The first vertex is always located at the midpoint of an edge.
- If the k-th point is located to the left of the $k-1$ previous points, its actual position is located at the midpoint of the segment defined by the left edge corner and the leftmost already assigned Steiner point. Similarly, if the k-th point is located to the right, its actual position is determined by the midpoint of the segment defined by the rightmost assigned Steiner point and the right edge corner.
- If the relative position of the k-th point is between two Steiner points already allocated, its actual position is determined by the midpoint of the two Steiner points.

Under this approach we identified 14 new co-spherical elements in the tessellations of 1-irregular cubes. A description of each of them can be found in Table 9.

Table 9. New co-spherical elements while tessellating 1-irregular cubes

Element	Vertices	Edges	Faces	Example
Generic #11	7	11	6	
Generic #12	7	11	6	
Generic #13	7	12	7	

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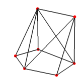
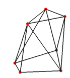
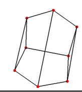
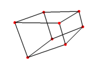
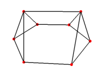
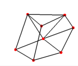
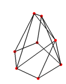
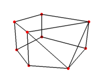
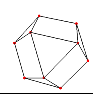
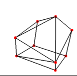
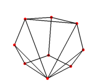
Element	**Vertices**	**Edges**	**Faces**	**Example**
Generic #14	7	13	8	
Generic #15	7	14	9	
Generic #16	8	12	6	
Generic #17	8	12	6	
Generic #18	8	13	7	
Generic #19	8	14	8	
Generic #20	8	15	9	
Generic #21	9	15	8	
Generic #22	9	15	8	
Generic #23	9	16	9	
Generic #24	9	16	9	

5.2 Evaluating the impact of each co-spherical element

Table 10 shows a summary of the results obtained by generating the tessellations of all 1-irregular configurations of a cube:

Table 10. Frequency of co-spherical elements while tessellating 1-irregular cubes

Element	Freq.	Element	Freq.
Cuboid	531	Generic #6	58
Tetrahedron	39,590,100	Generic #7	1,881
Quadrilateral Pyramid	5,200,926	Generic #8	2,340
Triangular Prism	184,374	Generic #9	6
Tetrahedron Comp.	11,220	Generic #10	108
Def. Prism	84,200	Generic #11	6,236
Def. Tetrahedron Comp.	14,701	Generic #12	9,288
Pentagonal Pyramid	171,838	Generic #13	3,972
Hexagonal Pyramid	7,353	Generic #14	874
Triangular Bipyramid	625,447	Generic #15	4,966
Quadrilateral Bipyramid	139,851	Generic #16	146
Pentagonal Bipyramid	25,686	Generic #17	204
Hexagonal Bipyramid	1,755	Generic #18	1,361
Biprism	148	Generic #19	4,033
Generic #1	61,044	Generic #20	197
Generic #2	186,594	Generic #21	42
Generic #3	94,020	Generic #22	214
Generic #4	28,218	Generic #23	4
Generic #5	28,028	Generic #24	6
		Total	**46,491,970**

The trend observed in the bisection based approach is also observed here: the most used elements are tetrahedra and quadrilateral pyramids (∼85.15% and ∼11.19% of total elements respectively, corresponding to more than 96%). The initial set of seven elements represents a ∼96.98%, while the set of 24 elements found in configurations under the bisection based approach covers a ∼99.93%. Finally, the elements that appear exclusively under the intersection based approach represent only a ∼0.07% of the total.

6 Conclusions

We have identified 24 co-spherical elements while tessellating 1-irregular cubes generated by a bisection based approach and 38 co-spherical elements while tessellating 1-irregular cubes generated by an intersection based approach. We have experimentally noticed that in the tessellation of 1-irregular cubes (aspect ratio equal to 1) more co-spherical elements appear than in the tessellation of 1-irregular cuboids with larger aspect ratio. When we increase the cuboid aspect ratio a subset of these co-spherical elements is required and no new co-spherical element appears.

We have studied the tessellations of 1-irregular cuboids generated by a bisection based approach with three different aspect ratios: 1, $\sqrt{2}$, and 4. The results can be summarized as follows:

- All the tessellations for 1-irregular cubes and 1-irregular cuboids with aspect ratio from 1 to $\sqrt{2}$ are adequate for the finite volume method. We would need to add 6 co-spherical elements to the initial final element set if we want that our mixed element mesh generator can tessellate the 1-irregular cuboids the first time the mesh is done 1-irregular.
- The number of different co-spherical elements while tessellating 1-irregular cubes can be reduced from 24 to 16 by adding internal faces and to 13 by adding internal faces and edges. While tessellating 1-irregular cuboids with aspect ratio equal to $\sqrt{2}$, the required elements are reduced from 10 to 7 if we allow the insertion of internal faces. While tessellating 1-irregular cuboids with aspect ratio equal to 4 only 6 co-spherical elements are used.

We have also study the tessellations of 1-irregular cubes generated by an intersection based approach and 14 additional co-spherical elements appear. They represent less than 0.07% of the total, then it not useful to include them in the set of final elements. They would increase this set in ∼58% (24 to 38). It is worth to point out that the proposed algorithm was only applied for tessellating 1-irregular cuboids but it can also be used without any modification to tessellate any 1-irregular convex configuration: 1-irregular prisms, pyramids or tetrahedra, among others. Moreover, the algorithm can be used to generate Delaunay tessellations for any point set. It may be only required to recognize new co-spherical configurations. This means we could apply this algorithm to the points of a larger part of the 1-irregular mixed element mesh and not only to the 1-irregular basic elements. The circumcenter requirement is only really necessary for 1-irregular elements that are located at the boundary or at a material interface.

We have made the study under the assumption that all the 1-irregular configurations appear in the same rate, but this is for sure not true. While generating a mesh based on modified octrees, there are some configurations that appear more frequently than others. This fact could mean that some co-spherical elements belonging to the tessellation of few 1-irregular cuboids, could have a greater impact than the one we have computed if these few configurations appear very frequently while generating a mesh. A complete study should consider also this case.

The study presented here is very useful for our mesh generator based on modified octrees, but we think that it can also be useful for other mesh generator based on octrees.

Acknowledgments

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